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## CONTENTS

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# A FEW THEOREMS ON GENERALISED LAPLACE TRANSFORM

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1. The object of this paper is to prove a few theorems on the generalised Laplace Transform defined by

$$f(p) = p \int_0^\infty e^{-\frac{1}{2}px} (px)^{m-\frac{1}{2}} W_{k,m}(px)\, h(x)\, dx \tag{1}$$

which we symbolically denote as

$$f(p) \overset{v}{\underset{v}{=}} h(x)$$

This transform is based on Dr. Varma's recent generalisation [10] of the Laplace's integral

$$f(p) = p \int_0^\infty e^{-px}\, h(x)\, dx. \tag{2}$$

Integral (2) will, as usual, be denoted by

$$f(p) \doteq h(x).$$

When $k = -m + \frac{1}{2}$, (1) reduces to (2) on account of the identity

$$W_{-m+\frac{1}{2}, m}(x) \equiv x^{-m+\frac{1}{2}} e^{-\frac{1}{2}x}.$$

2. THEOREM I. If

$$f_1(p) \doteqdot_{\nu}^{\nu} h_1(x)$$

and

$$f_2(p) \doteqdot_{\nu}^{\nu} h_2(x)$$

then

$$\int_0^\infty f_1(x) h_2(x) \frac{dx}{x} = \int_0^\infty f_2(x) h_1(x) \frac{dx}{x}.$$

*Proof.*—We have

$$f_1(p) = p \int_0^\infty (px)^{m-\frac{1}{2}} e^{-\frac{1}{2}px} W_{k,m}(px) h_1(x)\, dx$$

and

$$f_2(p) = p \int_0^\infty (px)^{m-\frac{1}{2}} e^{-\frac{1}{2}px} W_{k,m}(px) h_2(x)\, dx \tag{4}$$

Hence

$$\int_0^\infty f_1(x) h_2(x) \frac{dx}{x} = \int_0^\infty h_2(x)\, dx \int_0^\infty (xt)^{m-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k,m}(xt) h_1(t)\, dt$$

$$= \int_0^\infty h_1(t)\, dt \int_0^\infty (xt)^{m-\frac{1}{2}} e^{-\frac{1}{2}xt} W_{k,m}(xt) h_2(x)\, dx$$

on changing the order of integration,

$$= \int_0^\infty h_1(t) f_2(t) \frac{dt}{t}.$$

The change in the order of integration can easily be justified if (3) and (4) are absolutely convergent for which, if $h_1(x) = 0(x^{\mu_1})$ and $h_2(x) = 0(x^{\mu_2})$ for small $x$, we must have $R(\mu_1) > -1$, $R(\mu_1 + 2m) > -1$, $R(\mu_2) > -1$, $R(\mu_2 + 2m) > -1$ and $h_1(x) = 0(x^{\nu_1})$, $h_2 = 0(x^{\nu_2})$ for large $x$. We have, however, imposed hard conditions for justifying the change in the order of integration. These may be relaxed.

This theorem corresponds to Parseval-Goldstein theorem of Operational Calculus [1].

3. THEOREM II. If

$$f(p) \doteqdot h(x)$$

and

$$p^{m-k+\frac{1}{2}}\, h\left(\frac{1}{p}\right) \frac{\nu}{\overline{\nu}}\, g(x)$$

then

$$f(p) = 2p^{k+\frac{1}{2}} \int_0^\infty t^m K_{2m}(2\sqrt{pt})\, g(t)\, dt, \tag{5}$$

provided that the integral is convergent.

If, in addition,

$$F(p) \doteqdot x^{\frac{1}{2}+m-l}\, g(x),$$

then

$$f(p) = p_k^{-l} \int_0^\infty e^{-\frac{1}{2}u}\, u^{l-1}\, W_{l,\,m}(u)\, F\left(\frac{p}{u}\right) du \tag{6}$$

when this integral converges.

*Proof.*—Goldstein has shown that [1]

$$\int_0^\infty e^{-\frac{1}{2}u-\frac{pt}{u}}\, u_k^{-2}\, W_{k,\,m}(u)\, du = 2\,(pt)_k^{-\frac{1}{2}}\, K_{2m}(2\sqrt{pt}) \tag{7}$$

From this it follows that

$$e^{-\frac{a}{x}}\, x^{k-m-3/2} \frac{\nu}{\overline{\nu}}\, 2a_k^{-\frac{1}{2}}\, p^{m+1}\, K_{2m}(2\sqrt{ap})$$

Applying Theorem I to this and the relation

$$p^{m-k+\frac{1}{2}}\, h\left(\frac{1}{p}\right) \frac{\nu}{\overline{\nu}}\, g(x),$$

we have

$$2a^{k+\frac{1}{2}} \int_0^\infty x^m K_{2m}(2\sqrt{ax})\, g(x)\, dx$$

$$= a \int_0^\infty e^{-\frac{a}{x}}\, x^{-2}\, h\left(\frac{1}{x}\right) dx = f(a).$$

Next, using (7) in (5) we have

$$f(p) = p^{k+\frac{1}{2}} \int_0^\infty g(t)\, t^m (pt)^{\frac{1}{2}-l}\, dt \int_0^\infty e^{-\frac{pt}{u}-\frac{1}{2}u}\, u^{l-2}\, W_{k,m}(u)\, du$$

$$= p^{k-l} \int_0^\infty e^{-\frac{1}{2}u}\, u^{l-1}\, W_{l,m}(u)\, du \,.\, \frac{p}{u} \int_0^\infty e^{-\frac{p}{u}t}\, t^{m-l+\frac{1}{2}}\, g(t)\, dt,$$

on inverting the order of integration,

$$= p^{k-l} \int_0^\infty e^{-\frac{1}{2}u}\, u^{l-1}\, W_{l,m}(u)\, F\left(\frac{p}{u}\right) du$$

where

$$F(p) \doteqdot x^{m-l+\frac{1}{2}}\, g(x).$$

The change in the order of integration can be justified when $R(\mu + m - l + 3/2) > 0$ where $g(x) = 0(x^\mu)$ for small $x$ and $g(x) = 0(x^\nu)$ for large $x$ and the resulting integral is convergent.

When $k = -m + \frac{1}{2}$, this theorem yields known theorems of Operational Calculus [9], [3].

4. In this section we derive the generalised Laplace transforms of a few functions which we shall require later on.

(i) From the integral [8]

$$E(\alpha, \beta, \gamma : \delta : p) = \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\nu + \lambda - l + \frac{1}{2})}{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(2\nu + \lambda)}$$

$$\times \int_0^\infty e^{-\frac{1}{2}u}\, u^{\nu+\lambda-3/2}\, W_{l,\nu}(u)$$

$$\times \;{}_3F_3\left(\alpha, \beta, \gamma, \nu + \lambda - l + \tfrac{1}{2};\; \delta, \lambda, \lambda + 2m;\; -\frac{u}{p}\right) du$$

we easily derive that

$$x^{\lambda-1}\; {}_4F_3(\alpha, \beta, \gamma, \lambda + m - k + \tfrac{1}{2};\; \delta, \lambda, \lambda + 2m;\; -x)$$

$$\overset{\nu}{\underset{\nu}{=}} \frac{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda + 2m)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\lambda + m - k + \frac{1}{2})}\, E(\alpha, \beta, \gamma : \delta : p) \qquad (8)$$

$$R(\lambda) > 0,\; R(\lambda + 2m) > 0,\; R(p) > 0.$$

(ii) Pasricha [7] has shown that

$$\int_0^\infty x^{\mu-1} e^{-(a^2-\frac{1}{2})x} W_{k,m}(x) {}_rF_s\left(\begin{matrix}\beta_1, \beta_2, \ldots, \beta_r \\ \gamma_1, \gamma_2, \ldots, \gamma_r\end{matrix}; \nu x^\lambda\right) dx$$

$$= \sum_{n=0}^{\infty} \left\{\frac{(\beta_1)_n (\beta_2)_n \ldots (\beta_r)_n \, \Gamma(a+\lambda n)\, \Gamma(b+\lambda n)}{(\gamma_1)_n (\gamma_2)_n \ldots (\gamma_s)_n \, \Gamma(c+\lambda n)} \nu^n\right.$$

$$\left. \times\, {}_2F_1(a+\lambda n, b+\lambda n; c+\lambda n; -a^2)\right\}$$

where $a=\mu+m+\frac{1}{2}$, $b=\mu-m+\frac{1}{2}$, $c=\mu-k+1$.

From this we deduce after a little simplification that

$$x^{\nu-1} {}_rF_s(\beta_1, \beta_2, \ldots \beta_r; \gamma_1, \gamma_2, \ldots \gamma_s; \pm x^2)$$

$$\overset{\nu}{\underset{\nu}{=}} \frac{\Gamma(\nu)\,\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2})} p^{1-\nu}$$

$${}_{r+4}F_{s+2}\left\{\begin{matrix}\beta_1, \beta_2, \ldots \beta_r, \frac{1}{2}\nu, \frac{1}{2}(\nu+1), \frac{1}{2}(\nu+2m), \frac{1}{2}(\nu+2m+1); \\ \gamma_1, \gamma_2, \ldots, \gamma_s, \frac{1}{2}(\nu+m-k+\frac{1}{2}), \frac{1}{2}(\nu+m-k+3/2);\end{matrix} \pm \frac{4}{p^2}\right\} \tag{9}$$

$R(\nu)>0$, $R(\nu+2m)>0$, $\gamma<s-1$, $R(p)>0$.

If $\gamma=s-1$ then $R(p)>2$.

5. We now apply Theorem II to evaluate a few integrals.

(*a*) Starting with (8)

$$g(x) = x^{\lambda-1} {}_4F_3(\alpha, \beta, \gamma, \lambda+m-k+\tfrac{1}{2}; \delta, \lambda, \lambda+2m; -x)$$

$$\overset{\nu}{\underset{\nu}{=}} \frac{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda+2m)\, p^{1-\lambda}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\lambda+m-k+\frac{1}{2})} E(\alpha, \beta, \gamma : \delta : p)$$

$$= p^{m-k+\frac{1}{2}} h\left(\frac{1}{p}\right), \; R(\lambda)>0, \; R(\lambda+2m)>0, \; R(p)>0.$$

we have ([4], 255).

$$h(x) = \frac{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda+2m)\, x^{\lambda-k+m-\frac{1}{2}}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\lambda+m-k+\frac{1}{2})} E\left(\alpha, \beta, \gamma : \delta : \frac{1}{x}\right)$$

$$\doteqdot \frac{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda+2m)\, p^{k-m-\lambda+\frac{1}{2}}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\lambda+m-k+\frac{1}{2})} E(\alpha, \beta, \gamma, \lambda+m-k+\tfrac{1}{2} : \delta : p)$$

$$= f(p), \; R(\lambda+m-k+\tfrac{1}{2})>0.$$

Hence (5) gives on writing $k = \lambda + m - \mu + \frac{1}{2}$

$$E(\alpha, \beta, \gamma, \mu : \delta : p) = \frac{2\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}{\Gamma(\delta)\,\Gamma(\lambda)\,\Gamma(\lambda+2m)}\, p^{\lambda+m}$$

$$\times \int_0^\infty t^{m+\lambda-1} K_{2m}(2\sqrt{pt})\, {}_4F_3(\alpha, \beta, \gamma, \mu;\ \delta, \lambda, \lambda+2m;\ -t)\, dt \tag{10}$$

valid by analytic continuation for $R(\lambda) > 0$, $R(\lambda + 2m) > 0$, $R(p) > 0$.

When $\lambda = \mu$, this yields

$$E(\alpha, \beta, \gamma, \mu : \delta : p) = \frac{2\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\delta)\,\Gamma(\mu+2m)}\, p^{\mu+m}$$

$$\times \int_0^\infty t^{m+\mu-1}\, {}_3F_2(\alpha, \beta, \gamma;\ \delta, \mu+2m;\ -t)\, K_{2m}(2\sqrt{pt})\, dt \tag{11}$$

$$R(\mu) > 0,\ R(\mu+2m) > 0,\ R(p) > 0.$$

If in (10) we take $m = \frac{1}{4}$, $t = x^2$ replace $2\sqrt{p}$ by $p$ and use the result

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

we get, after a little simplification,

$$E(\alpha, \beta, \gamma, \mu : \delta : \tfrac{1}{4}p^2) = \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}{\Gamma(\delta)\,\Gamma(2\lambda)}\, p^{2\lambda-1}$$

$$\times\, p \int_0^\infty e^{-px}\, x^{2\lambda-1}\, {}_4F_3(\alpha, \beta, \gamma, \mu;\ \delta, \lambda, \lambda+\tfrac{1}{2};\ -x^2)\, dx$$

from which it follows that

$$x^{2\lambda-1}\, {}_4F_3(\alpha, \beta, \gamma, \mu;\ \delta, \lambda, \lambda+\tfrac{1}{2};\ -x^2)$$

$$\doteqdot \frac{\Gamma(\delta)\,\Gamma(2\lambda)\,p^{1-2\lambda}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}\, E(\alpha, \beta, \gamma, \mu : \delta : \tfrac{1}{4}p^2) \tag{12}$$

$$R(\lambda) > 0,\ R(p) > 0.$$

Now in (10) we take $\alpha = \frac{1}{2} - \chi - \nu$, $\beta = \frac{1}{2} - \chi + \nu$, $\gamma = \frac{1}{2} - \chi$, $\mu = 1 - \chi$, $\delta = 1 - 2\chi$, and use

$$E(\tfrac{1}{2} - k + m,\ \tfrac{1}{2} - k - m,\ \tfrac{1}{2} - k,\ 1 - k : 1 - 2k : \tfrac{1}{4}z^2)$$

$$= 2^{2k} \sqrt{k}\, \Gamma(\tfrac{1}{2} - k + m)\, \Gamma(\tfrac{1}{2} - k - m)\, z^{-2k}\, W_{k,m}(iz)\, W_{k,m}(-iz) \tag{13}$$

We then have

$$W_{\chi,\nu}(2i\sqrt{p})\; W_{\chi,\nu}(-2i\sqrt{p}) = \frac{2^{2\chi+1}\,p^{\lambda+m+\chi}}{\Gamma(\lambda)\,\Gamma(\lambda+2m)}$$

$$\times \int_0^\infty t^{m+\lambda-1} K_{2m}(2\sqrt{pt})\; {}_4F_3\left(\begin{matrix}\frac{1}{2}-\chi-\nu,\ \frac{1}{2}-\chi+\nu,\ \frac{1}{2}-\chi,\ 1-\chi;\\ 1-2\chi,\ \lambda,\ \lambda+2m;\end{matrix}\ -t\right) dt$$

$$R(\lambda) > 0,\ R(\lambda+2m) > 0,\ R(p) > 0. \tag{14}$$

This result has been given by Meijer [6].

(*b*) Taking (12)

$$h(x) = x^{2\lambda-1}\,{}_4F_3(\alpha,\beta,\gamma,\mu;\ \delta,\lambda,\lambda+\tfrac{1}{2};\ -\tfrac{1}{4}x^2)$$

$$\doteqdot \frac{\Gamma(\delta)\,\Gamma(2\lambda)\,p^{1-2\lambda}}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}\ E(\alpha,\beta,\gamma,\mu:\delta:p^2)$$

$$= f(p),\ R(\lambda) > 0,\ R(p) > 0,$$

we have

$$p^{m-k+\frac{1}{2}}\, h\left(\frac{1}{p}\right) = p^{m-k-2\lambda-3/2}\,{}_4F_3\left(\alpha,\beta,\gamma,\mu;\ \lambda,\lambda+\tfrac{1}{2};\ -\frac{1}{4p^2}\right)$$

$$\overset{\nu}{\underset{\nu}{=}}\ \frac{x^{2\nu-1}\,\Gamma(2\lambda)}{\Gamma(2\nu)\Gamma(2\nu+2m)}.$$

$$\times\ {}_4F_5\left(\alpha,\beta,\gamma,\mu;\delta,\ \nu,\ \nu+\tfrac{1}{2},\ \nu+m,\ \nu+m+\tfrac{1}{2};-\frac{x^2}{16}\right)$$

$$= g(x),\ R(\nu) > 0,\ R(\nu+m) > 0,\ 2\nu = 2\lambda+k-m+\tfrac{1}{2},$$

on backward interpretation with the help of (9) and using the relation: If

$$f(p)\ \overset{\nu}{\underset{\nu}{=}}\ h(x)\ \text{then}\ f(ap)\ \overset{\nu}{\underset{\nu}{=}}\ h\left(\frac{x}{a}\right),\ a > 0.$$

Applying (5) we get

$$E(\alpha,\beta,\gamma,\mu:\delta:p^2) = \frac{2\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}{\Gamma(\delta)\,\Gamma(2\nu)\,\Gamma(2\nu+2m)}\,p^{\nu+m}$$

$$\times \int_0^\infty t^{m+2\nu-1} K_{2m}(2\sqrt{pt})$$

$$\times\ {}_4F_5\left\{\begin{matrix}\alpha,\beta,\gamma,\mu;\\ \delta,\ \nu,\ \nu+\frac{1}{2},\ \nu+m,\ \nu+m+\frac{1}{2};\end{matrix}\ -\frac{t^2}{16}\right\} dt \tag{15}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(p) > 0.$$

In particular, if we take $\alpha = \frac{1}{2} - \xi - \eta$, $\beta = \frac{1}{2} - \xi + \eta$, $\gamma = \frac{1}{2} - \xi$, $\mu = 1 - \xi$, $\delta = 1 - 2\xi$ and use (13) we get a known result ([2], 113).

Also

$$x^{\frac{1}{2}+m-l}\, g(x) = \frac{\Gamma(2\lambda)}{\Gamma(2\nu)\,\Gamma(2\nu+2m)}\, x^{2\nu+m-l-\frac{1}{2}}$$

$$\times\ {}_4F_5\left(\alpha, \beta, \gamma, \mu;\ \delta, \nu, \nu+\tfrac{1}{2}, \nu+m, \nu+m+\tfrac{1}{2};\ -\frac{x^2}{16}\right)$$

$$\doteqdot \frac{\Gamma(2\nu+m-l+\frac{1}{2})\,\Gamma(2\lambda)}{\Gamma(2\nu)\,\Gamma(2\nu+2m)}\, p^{\frac{1}{2}+l-2\nu-m}$$

$${}_6F_5\left\{\begin{matrix}\alpha, \beta, \gamma, \mu, \frac{1}{4}+\nu+\frac{1}{2}(m-l), \frac{3}{4}+\nu+\frac{1}{2}(m-l); \\ \delta, \nu, \nu+\frac{1}{2}, \nu+m, \nu+m+\frac{1}{2};\end{matrix}\ -\frac{1}{4p^2}\right\}$$

$$= F(p),\quad R(2\nu+m-l+\tfrac{1}{2}) > 0.$$

Hence (6) gives

$$E(\alpha, \beta, \gamma, \mu : \delta : p^2) = \frac{\Gamma(2\nu+m-l+\frac{1}{2})\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}{\Gamma(\delta)\,\Gamma(2\nu)\,\Gamma(2\nu+2m)}$$

$$= \int_0^\infty e^{-\frac{1}{2}u}\, u^{2\nu+m-3/2}\, W_{l,m}(u)$$

$$\times\ {}_6F_5\left\{\begin{matrix}\alpha, \beta, \gamma, \mu, \frac{1}{4}+\nu+\frac{1}{2}(m-l), \frac{3}{4}+\nu+\frac{1}{2}(m-l); \\ \delta, \nu, \nu+\frac{1}{2}, \nu+m, \nu+m+\frac{1}{2};\end{matrix}\ -\frac{u^2}{4p^2}\right\} du \tag{16}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(p) > 0.$$

Now taking $\alpha = \frac{1}{2} - \xi - \eta$, $\beta = \frac{1}{2} - \xi + \eta$, $\gamma = \frac{1}{2} - \xi$, $\mu = 1 - \xi$, $\delta = 1 - 2\xi$ and replacing $2p$ by $z$ we get by virtue of (13)

$$W_{\xi,\eta}(iz)\, W_{\xi,\eta}(-iz) = \frac{\Gamma(2\nu+m-l+\frac{1}{2})\, z^{2\xi}}{\Gamma(2\nu)\,\Gamma(2\nu+2m)}$$

$$\times \int_0^\infty e^{-\frac{1}{2}u}\, u^{2\nu+m-3/2}\, W_{l,m}(u)$$

$$\times\ {}_6F_5\left\{\begin{matrix}\frac{1}{2}-\xi-\eta, \frac{1}{2}-\xi+\eta, \frac{1}{2}-\xi, 1-\xi, \frac{1}{4}+\nu+\frac{1}{2}(m-l), \frac{3}{4}+\nu+\frac{1}{2}(m-l); \\ 1-2\xi, \nu, \nu+\frac{1}{2}, \nu+m, \nu+m+\frac{1}{2};\end{matrix}\ -\frac{u^2}{z^2}\right\} du \tag{17}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(z) > 0.$$

If we take $\alpha = \frac{1}{2} - \xi - \eta$, $\beta = \frac{1}{2} - \xi + \eta$, $\gamma = 1$, $\mu = \delta$ and apply

$$E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m, 1 :: x) = \Gamma(\tfrac{1}{2}-k+m)\,\Gamma(\tfrac{1}{2}-k-m) \times (4x)^{\frac{1}{2}-k}\, S_{2k,2m}(2\sqrt{x}) \tag{18}$$

we get from (15) and (16) respectively the results

$$S_{2\xi,\,2\eta}(2p) = \frac{2^{2\xi}\, p^{2\nu+m+2\xi-1}}{\Gamma(2\nu)\,\Gamma(2\nu+2m)}$$

$$\times \int_0^\infty t^{2\nu+m-1} K_{2m}(2\sqrt{pt})$$

$$\times\ {}_3F_4\left(\begin{matrix}\frac{1}{2}-\xi-\eta,\ \frac{1}{2}-\xi+\eta,\ 1; \\ \nu,\ \nu+\frac{1}{2},\ \nu+m,\ \nu+m+\frac{1}{2};\end{matrix}\ -\frac{t^2}{16}\right) dt. \tag{19}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(p) > 0$$

and

$$S_{2\xi,\,2\eta}(2p) = \frac{(2p)^{2\xi-1}\,\Gamma(2\nu+m-l+\frac{1}{2})}{\Gamma(2\nu)\,\Gamma(2\nu+2m)}$$

$$\times \int_0^\infty e^{-\frac{1}{2}u}\, u^{2\nu+m-3/2}\, W_{l,\,m}(u)$$

$$\times\ {}_5F_4\left\{\begin{matrix}\frac{1}{2}-\xi-\eta,\ \frac{1}{2}-\xi+\eta,\ 1,\ \frac{1}{4}+\nu+\frac{1}{2}(m-l),\ \frac{3}{4}+\nu+\frac{1}{2}(m-l); \\ \nu,\ \nu+\frac{1}{2},\ \nu+m,\ \nu+m+\frac{1}{2};\end{matrix}\ -\frac{u^2}{4p^2}\right\} du \tag{20}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(p) > 0.$$

From (16) we also infer that

$$x^{2\nu-1}\ {}_6F_5\left\{\begin{matrix}\alpha,\ \beta,\ \gamma,\ \mu,\ \frac{1}{4}+\nu+\frac{1}{2}(m-k),\ \frac{3}{4}+\nu+\frac{1}{2}(m-k); \\ \delta,\ \nu,\ \nu+\frac{1}{2},\ \nu+m+\frac{1}{2};\end{matrix}\ -x^2\right\}$$

$$\overset{\nu}{\underset{\nu}{=}} \frac{\Gamma(\delta)\,\Gamma(2\nu)\,\Gamma(2\nu+2m)\,p^{1-2\nu}}{\Gamma(2\nu+m-k+\frac{1}{2})\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\mu)}$$

$$\times E(\alpha, \beta, \gamma, \mu : \delta : \tfrac{1}{4}p^2) \tag{21}$$

$$R(\nu) > 0,\ R(\nu+m) > 0,\ R(p) > 0.$$

6. THEOREM III. If

$$f(p) \overset{\nu}{\underset{\nu}{=}} h(x)$$

and

$$p^{k-m-\frac{1}{2}}\, h\left(\frac{1}{p}\right) \doteqdot g(x),$$

then

$$f(p) = 2p^{m+1} \int_0^\infty t^{k-\frac{1}{2}} K_{2m}(2\sqrt{pt})\, g(t)\, dt \tag{22}$$

and

$$f(p) = p^{m-l+\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}u} u^{l-1} W_{l,m}(u) F\left(\frac{p}{u}\right) du \tag{23}$$

where

$$F(p) \doteqdot x^{k-l} g(x).$$

(22) follows on applying Parseval-Goldstein theorem to the relations

$$p^{k-m-\frac{1}{2}} h\left(\frac{1}{p}\right) \doteqdot g(x)$$

and ([5], 51)

$$2\sqrt{a}\, p^{k+\frac{1}{2}} K_{2m}(2\sqrt{ap}) \doteqdot x^{-k} e^{-\frac{a}{2x}} W_{k,m}\left(\frac{a}{x}\right)$$

whilst (23) is obtained by the process employed for (6).

My best thanks are due to Dr. R. S. Varma for his guidance and interest in the preparation of this paper.

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# ON A SPECIAL CASE OF TWO-DIMENSIONAL FLOW

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THE object of this note is to study a special class of two-dimensional self-superposable flows. The potential flow

$$u = -ax, \; v = ay$$

represents the well-known flow at a stagnation point, and the problem of finding the flow in the vicinity of a stagnation point for a viscous fluid having the plane $x = 0$ as a fixed wall with $u = v = 0$, at the wall, but such that at large distances from it the motion is essentially the same as that given by the above potential flow, has been completely solved.[1]

In this paper we study the two-dimensional motion given by

$$u = -any^{n-1} f(x)$$

$$v = ay^n f'(x),$$

from the point of view of self-superposability, $a$ being a function of $t$.

The condition of continuity is thus automatically satisfied. We have

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = ay^n f''(x) + an(n-1)\, y^{n-2} f(x)$$

substituting in the equation of self-superposability

$$\left(u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = 0\right)^2,$$

$$-any^{n-1} f(x) \{ay^n f'''(x) + an(n-1)\, y^{n-2} f'(x)\}$$

$$+ ay^n f'(x) \{any^{n-1} f''(x) + an(n-1)(n-2)\, y^{n-3} f(x)\} = 0$$

we must therefore have

$$f(x).f'''(x) - f'(x).f''(x) = 0 \tag{1}$$

and

$$n^2 (n-1) - n(n-1)(n-2) = 0, \quad i.e., \quad n = 0, 1$$

Three solutions of (1) are

$$f(x) = A_1\, x,\ A \sin(\lambda x + B),\ ke^{-px},\ \left.\begin{matrix} A_1,\ A,\ \lambda \\ B,\ k,\ p \end{matrix}\right\}\ \text{being all constants.}$$

Of these $f(x) = A_1\, x$ gives the above potential flow, *viz.*, $u = -\,ax$, $v = ay$ for $n = 1$, $n = 0$ being trivial.

$$\text{I.}\quad f(x) = A \sin(\lambda x + B) \text{ with } n = 1 \tag{2}$$

In a non-viscous liquid $\alpha$ must be a constant,[3] and we have

$$u = -\, A\alpha \sin(\lambda x + B)$$

$$v = \alpha A \lambda y \cos(\lambda x + B)$$

The boundaries may be $y = 0$ or $x =$ constant, such that $\sin(\lambda x + B) = 0$. The solution is not valid for large values of $y$ as this makes $v$ large.

In a viscous fluid we have yet to satisfy the equation[4]

$$\frac{\partial \zeta}{\partial t} = \nu \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)$$

substituting for $\zeta$,

$$\frac{d\alpha}{dt} \cdot y f''(x) = \nu \alpha y f''''(x)$$

using (2),

$$\frac{d\alpha}{dt} = -\, \lambda^2 \nu \alpha, \quad i.e., \quad \alpha = \beta\, e^{-\nu \lambda^2 t}\ , \ \beta \text{ being a constant.}$$

We thus get

$$\left.\begin{aligned} u &= -\, \beta\, e^{-\nu \lambda^2 t} \sin(\lambda x + B) \\ v &= \lambda \beta\, e^{-\nu \lambda^2 t} y \cos(\lambda x + B) \end{aligned}\right\} \tag{3}$$

the constant $A$ being absorbed in $\beta$.

The motion decays exponentially with time, the rate of decay depending upon the kinematic viscosity of the fluid. For large values of $y$, $v$ becomes large, so that the solution is restricted to small values of $y$. If $y = 0$ is a boundary, the motion of the boundary is given by

$$u_0 = -\, \beta\, e^{-\nu \lambda^2 t} \sin(\lambda x + B),$$

*i.e.*, the boundary executes a damped hormonic motion. The stream function $\psi$ of the flow represented by (3) is given by

$$\psi = \beta y e^{-\nu\lambda^2 t} \sin(\lambda x + B).$$

The stream lines are therefore the cosecant curves.

We next show that an irrotational flow cannot be superimposed upon the flow represented by (3).

*Proof.*—Let $u_2$, $v_2$ be the velocity components of the irrotational flow superposable on (3). The equation of superposability of $(u_2, v_2)$ and $(u, v, \zeta)$ then becomes[5]

$$u_2 \lambda y \cos(\lambda x + B) + v_2 \sin(\lambda x + B) = 0.$$

Introducing the stream function $\psi$ of the irrotational flow the above equation can be written as

$$-\frac{\partial\psi}{\partial y}\lambda y \cos(\lambda x + B) + \frac{\partial\psi}{\partial x}\sin(\lambda x + B) = 0$$

of which the general solution is

$$\psi = f\{y \sin(\lambda x + B)\},$$

$f$ being an arbitrary function.

This value of $\psi$ has further to satisfy the condition $\nabla^2\psi = 0$, since the motion is irrotational. We have

$$\nabla^2\psi = f''\cdot\{\sin^2(\lambda x + B) + \lambda^2 y^2 \cos^2(\lambda x + B)\} - y\lambda^2 f'.\sin(\lambda x + B) = 0$$

Hence

$$\frac{f''}{f'} = \frac{\lambda^2 y \sin(\lambda x + B)}{\sin^2(\lambda x + B) + \lambda^2 y^2 \cos^2(\lambda x + B)}$$

The right-hand side must therefore be a function of $y \sin(\lambda x + B)$ which is not possible unless $\lambda = 0$ which makes the flow given by (3) irrotational and trivial.

II. We next take the case

$$f(x) = A \sin(\lambda x + B),\ n = 0$$

so that

$$u = 0,\ v = \alpha A\lambda \cos(\lambda x + B),\ \zeta = -\alpha A\lambda^2 \sin(\lambda x + B).$$

In the case of a viscous fluid $\alpha$ is again equal to $\beta e^{-\lambda^2\nu t}$, $\beta$ being a constant. So we have

$$u = 0, \; v = \beta \, e^{-\nu\lambda^2 t} \cos(\lambda x + \beta)$$

$\lambda$ and $A$ being absorbed in $\beta$.

The motion again decays exponentially with time. The boundaries for viscous flow are $\chi = \frac{1}{\lambda}\left(\frac{n\pi}{2} - B\right)$, $(n = 1, 3, 5 \ldots.)$. The motion is taking place between parallel walls. Here $v$ is always finite vanishing for large values of $t$.

III. We now take the case

$$f(x) = k \, e^{-px}, \; n = 1, \; k \text{ and } p \text{ being constants.}$$

Then

$$u = -\, \alpha k \, e^{-px}, \; v = -\, \alpha ykp \, e^{-px}, \; \zeta = \alpha ykp^2 \, e^{-px}$$

In a non-viscous liquid $y = 0$ is a boundary and the liquid is at rest at infinite distances from the $y$-axis. The solution is not valid for large values of $y$ which make $v$ large. This is true for positive values of $\chi$ only.

In viscous liquids we have again to satisfy.[6]

$$\frac{\partial \zeta}{\partial t} = v \nabla^2 \zeta$$

$$i.e., \frac{d\alpha}{dt} = v \, p^2 \alpha$$

so that $\alpha = \gamma \, e^{\nu p^2 t}$, $\gamma$ being a constant.

We have therefore

$$u = -\, \gamma \, e^{\nu p^2 t} \,.\, k \, e^{-px}, \; v = -\, \gamma \, e^{\nu p^2 t} \, ypk \, e^{-px}.$$

This solution is not of any physical interest as the velocities increase with time for finite values of $x$ and $y$.

My thanks are due to Dr. Ram Ballabh for his guidance and the Scientific Research Committee, U.P., for the monetary help given to me.

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# ON THE ABSORPTION SPECTRUM OF ZnTe MOLECULE IN THE VISIBLE REGION

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A band system of the ZnTe molecule occurring in the absorption in the region λλ 5662–4986, is reported in this paper. Mathur (1937) obtained continuous absorption for longer wavelength limits at λ 5608 in the visible, and at λ 3178 in the ultraviolet region. No absorption band or line has been recorded in the absorption spectrum by Mathur. The band heads are analysed, and found to conform to the formula,

$$\nu = 19616 \cdot 18 + (149 \cdot 0\, u' - 2 \cdot 0\, u'^2) - (384 \cdot 8\, u'' - 0 \cdot 9\, u''^2)$$

where $u = v + \frac{1}{2}$.

## Introduction

Besides Mathur's[1] data on the absorption on this molecule, no other information was available. He obtained continuous absorption with continuum limits at λ 5608 and λ 3176 located with the help of microphotograms. From these limits, he estimated the atomic term differences of Te in the excited and normal states in electron volts and calculated in K.Cals, the latent heat of this compound. No band or line has been recorded on his plates.

Mathur had used a 40-watt coiled coil bulb as the source for continuum, and a constant deviation spectrograph for studying the absorption spectrum, using Panchromatic plates for recording the spectrogram. The authors have obtained a well-defined band system in absorption in the region λλ 5662–4986. It is interesting to note that the longer wavelength limit of continuous absorption reported by Mathur, is exactly where the band system of the molecule begins. It is a matter of common experience as gained by the authors, in their studies of absorption spectra, that when a molecule gives a system of bands, the continuum of the background light is sharply cut off exactly, where the system begins, if the absorbing column of vapour is dense. Unless the absorbing column of vapour thins down the system of bands is not developed. Sometimes a reported continuous absorption of a molecule may after a careful manipulation of the absorbing column of vapour give rise to a well-defined system of bands.

## EXPERIMENTAL

The experimental arrangement was essentially the same as used by the first author[5] in his study of the absorption spectra of BiCl, BiS, SbS, and of other similar molecules by Sharma[2] (1950).

The ZnTe vapour was produced by heating a pure chemical prepared by Darmsdadt Company of Germany (available in very small quantity from a very old stock of chemicals of the Spectroscopic Department) inside a silica tube within a graphite furnace which was heated electrically to temperatures from 950° C. to 1275° C. The temperature was measured by an optical pyrometer of the vanishing filament type. The spectrogram is taken on the big Adam Hilger's glass prism spectrograph with a good light gathering and resolving power with a dispersion of 30 A/mm. An iron arc was used as a comparison spectrum and a 60-watt coiled bulb as the source for the continuum. Panchromatic plates were used to record the spectrograms. The system of bands developed between temperatures 1250° C. to 1275° C., and exposures ranging from 30 seconds to 8 minutes were repeated. Shorter exposures were needed for bringing out bands of the longer wavelength side, and longer exposures for those of the shorter wavelength side. It is significant that these bands appear to develop, when the furnace is kept filled with nitrogen at a slightly higher pressure than the usual value, *i.e.*, 30 to 40 cm. of Hg.

## RESULTS

In Table I are given the observed and calculated wave-numbers of the band heads in cm.$^{-1}$ in vacuum, the measured wavelengths in air in International Angstrom Units, their estimated intensities and vibrational assignments.

Table II gives the Deslander's arrangement of the observed band heads. The intensity plots lie on an open Condon parabola on account of large difference in the constants $\omega'_e$ and $\omega''_e$.

The band heads of the system appear to fit in the formula within the experimental error of about 6 cm.$^{-1}$ between the observed and calculated values.

$$\nu = 19616 \cdot 18 + 149 \cdot 0\, u' - 2 \cdot 0\, u'^2 - 384 \cdot 8\, u'' + 0 \cdot 9\, u''^2.$$

## DISCUSSION

Rosen[3] (1927) recorded in absorption an extensive system for $Te_2$ molecule $\lambda\lambda$ 3831·5–5664·5 A.U. Between $\lambda$ 5021 A.U. and 5664·5 A.U. Rosen has listed no less than 30 band heads of which seven alone are in agreement

TABLE I

| λ A.U. in air | Intensity | ν cm.$^{-1}$ to nearest whole number | | Assignment $v'$, $v''$ |
|---|---|---|---|---|
| | | Observed | Calculated | |
| 5662·5 | 1 | 17655 | 17650 | 3, 6 |
| 5590·2 | 3 | 17884 | 17887 | 2, 5 |
| 5515·2 | 5 | 18127 | 18122 | 1, 4 |
| 5474·5 | 3 | 18262 | 18263 | 2, 4 |
| 5447·4 | 1 | 18351 | 18354 | 0, 3 |
| 5406·3 | 0 | 18493 | 18499 | 1, 3 |
| 5363·8 | 0 | 18641 | 18640 | 2, 3 |
| 5295·0 | 0 | 18881 | 18879 | 1, 2 |
| 5256·2 | 8 | 19020 | 19020 | 2, 2 |
| 5219·2 | 6 | 19155 | 19157 | 3, 2 |
| 5191·8 | 10 | 19256 | 19260 | 1, 1 |
| 5153·1 | 8 | 19401 | 19401 | 2, 1 |
| *5124·1 | 0 | 19510 | .. | ? |
| 5116·5 | 7 | 19538 | 19538 | 3, 1 |
| 5081·6 | 4 | 19673 | 19671 | 4, 1 |
| 5052·3 | 0 | 19788 | 19784 | 2, 0 |
| 5018·5 | 3 | 19921 | 19921 | 3, 0 |
| 4986·0 | 0 | 20051 | 20054 | 4, 0 |

* Position not visible under the comparator, does not fit in the analysis.

TABLE II

| $v'$ \ $v''$ | 0 | 2 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 4 | 20051 | 19673 | | | | | |
| 3 | 19921 | 19538 | 19155 | | | | 17655 |
| 2 | 19788 | 19401 | 19020 | 18641 | 18262 | 17884 | |
| 1 | | 19256 | 18881 | 18493 | 18127 | | |
| 0 | | | | 18354 | | | |

with the band heads in Table I. The sequence, as well as the order, and their relative intensities of the bands as listed by Rosen, do not agree with those recorded by the authors. Further some of the strong band heads as in Table I, *e.g.*, λλ 5406–5153 are not recorded by Rosen. Olsson[4] (1935) recorded $Te_2$ bands in absorption between λλ 3913·5–4849·0 A.U.

Authors failed to observe bands of $Te_2$ in absorption under different conditions of temperatures, and exposure beyond the longer wavelength limit $\lambda$ 4957·6. ~~These considerations seem to throw doubt on the extensive system of $Te_2$ bands reported by Rosen.~~ It is therefore very likely that the system of bands recorded by the authors by vapourising a pure sample of ZnTe is due to ZnTe molecule. The constants $\omega_e$, $\omega_e x_e$, for $Te_2$ molecule as obtained by Rosen are entirely different from those obtained by the authors for ZnTe molecule. Authors have therefore assigned the system due to ZnTe molecule, though a further confirmation is needed by observing its absorption in the ultraviolet region.

The authors have great pleasure in recording their sincere thanks to Dr. K. Majumdar, D.Sc., for his guidance, and to Dr. D. Sharma, M.Sc., D.Phil., for his interest and valuable suggestions in this brief investigation.

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# ON THE NON-SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES

By S. R. Sinha

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(Communicated by Dr. B. N. Prasad)

1. Let $f(\theta)$ be a periodic function with period $2\pi$ and integrable-$L$ in $(-\pi, \pi)$. Let the fourier series of $f(\theta)$ be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{1.1}$$

Then the conjugate series of the fourier series is given by

$$\sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta). \tag{1.2}$$

and the conjugate function, defined as a Cauchy integral, is given by

$$g(\theta) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi(u) \cot \tfrac{1}{2}u \, du. \tag{1.3}$$

where

$$\psi(u) = f(\theta + u) - f(\theta - u).$$

2. In what follows we use the following notations:

$$\left.\begin{aligned} \Psi_\alpha(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u)\, du, && \alpha > 0; \\ \Psi_0(t) &= \psi(t); \\ \psi_\alpha(t) &= \Gamma(\alpha+1)\, t^{-\alpha}\, \Psi_\alpha(t), && \alpha \geqslant 0. \end{aligned}\right\} \tag{2.1}$$

$$\left.\begin{aligned} \Delta &= 1 - 2x\cos t + x^2 \\ Q'(t) &= \frac{x \sin t}{\Delta} \\ Q(t) &= \frac{t}{(p^2 + t^2)}, \\ &\text{where } p = \log x, \\ q(t) &= \frac{1}{t} - Q(t). \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} Q^{(p)}(t) &= \frac{d^p}{dt^p}\left\{Q(t)\right\} \\ q^{(p)}(t) &= \frac{d^p}{dt^p}\{q(t)\} \end{aligned}\right\} \tag{2.3}$$

$$\epsilon = \text{arc}\sin(1-x). \tag{2·4}$$

3. Plessner[1] had proved the following theorem:

THEOREM.—Let

$$V(x, \theta) = \sum_{n=1}^{\infty}(b_n \cos n\theta - a_n \sin n\theta)\, x^n, \qquad (0 \leqslant x < 1).$$

Then, if for any $\theta$, the condition

$$\int_0^t \psi(t)\, dt = o(t), \qquad (t \to 0) \tag{3.1}$$

is satisfied, then

$$\lim_{x \to 1}\left[V(x, \theta) - \frac{1}{2\pi}\int_{\epsilon}^{\pi} \psi(t) \cot \tfrac{1}{2}t\, dt\right] = 0.$$

It follows immediately from this theorem that if condition (3.1) holds then the divergence to $+\infty(-\infty)$ of the conjugate function (1.3) is a necessary and sufficient condition for the divergence of the Abel-limit of the conjugate series to $+\infty(-\infty)$.

Prasad[2] had obtained a more general theorem which included Plessner's theorem as a particular case. In Prasad's theorem the conjugate function was replaced by his Generalised Conjugate Function and the condition (3.1) was replaced by the less stringent condition, *viz.*,

$$\int_0^t \frac{\Psi(t)}{t}\, dt = o(t).$$

4. The object of the present paper is to generalize Plessner's theorem in another direction, *viz.*, by replacing the condition (3.1) by the less stringent condition

$$\psi_\alpha(t) = o(1), \text{ as } t \to 0, \tag{4.1}$$

$\alpha$ being any positive integer $\geqslant 1$.

We prove the following theorems:

THEOREM 1. If $\psi_\alpha(t) = o(1)$, as $t \to 0$, $\alpha$ being a positive integer $\geqslant 1$, then

$$\left[V(x, \theta) - \frac{1}{2\pi} \int_{\epsilon}^{\pi} \psi_{\alpha-1}(t) \cot \tfrac{1}{2}t \, dt\right] = o(1).$$

THEOREM 2. If

$$\psi_\alpha(t) = O(1), \text{ as } t \to 0, \ \alpha \text{ being a positive integer } \geqslant 1,$$

then

$$\left[V(x, \theta) - \frac{1}{2\pi} \int_{\epsilon}^{\pi} \psi_{\alpha-1}(t) \cot \tfrac{1}{2}t \, dt\right] = O(1). \qquad (4.2)$$

THEOREM 3. If

$$\psi_\alpha(t) = O(1), \text{ as } t \to 0, \ \alpha \text{ being a positive integer } \geqslant 1, \qquad (4.3)$$

then the divergence of the generalised conjugate function

$$\frac{1}{2\pi} \int_{\epsilon}^{\pi} \psi_{\alpha-1}(t) \cot \tfrac{1}{2}t \, dt$$

to $+\infty(-\infty)$ is a necessary and sufficient condition for the divergence to $+\infty(-\infty)$ of the Abel-limit of the conjugate series of a fourier series.

5. We shall require following lemmas,

*Lemma* 1.—

$$Q^{(r)}(t) = O\left(\frac{1}{t^{r+1}}\right), \quad (t \text{ large})$$

$$t^r Q^{(r-1)}(t) = O\left\{\left(\frac{t^2}{p^2}\right)^r\right\}, \quad (t \to 0)$$

*Lemma* 2.—

$$t^r q^{(r)}(t) = O\left(\frac{p^2}{t^3}\right).$$

The proof of these is obvious.

6. *Proof of Theorem* 1—We have

$$V(x, \theta) = \sum_{1}^{\infty} B_n(\theta)\, x^n.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta + t)\, Q'(x, t)\, dt.$$

$$= \frac{1}{\pi} \int_0^{\pi} \psi(\theta, t)\, Q'(x, t)\, dt.$$

$$= \frac{1}{\pi} \int_0^{\infty} \psi(t)\, Q(t)\, dt.$$

$$V(x, \theta) - \frac{1}{\pi} \int_{\epsilon}^{\infty} \frac{\psi_{\alpha-1}(t)}{t}\, dt$$

$$= \frac{1}{\pi} \int_0^{\infty} \psi(t)\, Q(t)\, dt - \frac{1}{\pi} \int_{\epsilon}^{\infty} \frac{\psi_{\alpha-1}(t)}{t}\, dt,$$

$$= I, \text{ say}$$

We need only prove that $I = o(1)$.

Integrating by parts $(\alpha - 1)$ times

$$\int_0^{\infty} \psi(t)\, Q(t)\, dt$$

$$= \left[ \sum_{\rho=1}^{\alpha-1} (-1)^{\rho-1}\, \Psi_{\rho}(t)\, Q^{(\rho-1)}(t) \right]_0^{\infty} + (-1)^{\alpha-1} \int_0^{\infty} \Psi_{\alpha-1}(t)\, Q^{(\alpha-1)}(t)\, dt,$$

$$= \left[ \sum_{\rho=1}^{\sigma-1} o(t^{\rho})\,.\, 0\left(\frac{1}{t}\, \rho\right) \right] + \frac{(-1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} \psi_{\alpha-1}(t)\, t^{\alpha-1}\, Q^{(\alpha-1)}(t)\, dt,$$

by Lemma 1.

$$= o(1) + \frac{(-1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} \psi_{\alpha-1}(t)\,.\, t^{\alpha-1}\, Q^{(\alpha-1)}(t)\, dt.$$

Thus

$$\pi.\, I = o(1) + \frac{(-1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\epsilon} \psi_{\alpha-1}(t)\,.\, t^{\alpha-1} Q^{(\alpha-1)}(t)\, dt$$

$$+ \frac{(-1)^{\alpha}}{\Gamma(\alpha)} \int_{\epsilon}^{\infty} \psi_{\alpha-1}(t)\,.\, t^{\alpha-1} q^{(\alpha-1)}(t)\, dt.$$

$$= o(1) + \frac{(-1)^{\alpha-1}}{\Gamma(\alpha)}\, I_1 + \frac{(-1)^{\alpha}}{\Gamma(\alpha)}\, I_2, \text{ say}$$

Now

$$I_1 = \Big[\psi_\alpha(t).t^\alpha.Q^{(\alpha-1)}(t)\Big]_0^\epsilon - (\alpha-1)\int_0^\epsilon \psi_\alpha(t).t^{\alpha-1}.Q^{(\alpha-1)}(t)\,dt$$

$$-\int_0^\epsilon \psi_\alpha(t).t^\alpha.Q^{(\alpha)}(t)\,dt,$$

$$= o(1).0\left[\left(\frac{\epsilon^2}{p^2}\right)^\alpha\right] + o\left[\frac{1}{p^{2\alpha}}\int_0^\epsilon t^{2\alpha-1}dt\right] + o\left[\frac{1}{p^{2\alpha+2}}\int_0^\epsilon t^{2\alpha+1}\,dt\right],$$

$$= o(1).$$

Next

$$I_2 = \int_\epsilon^\infty \psi_{\alpha-1}(t).t^{\alpha-1}q^{(\alpha-1)}(t)\,dt$$

$$= \Big[\psi_\alpha(t).t^\alpha.q^{(\alpha-1)}(t)\Big]_\epsilon^\infty - \int_\epsilon^\infty \psi_\alpha(t)\Big[(\alpha-1)\,t^{\alpha-1}q^{(\alpha-1)}(t) + t^\alpha q^{(\alpha)}(t)\Big]\,dt,$$

$$= o(1)\left\{t^\alpha.0\left(\frac{1}{t^\alpha}\right)\right\}_{t\to\infty} + o(1).0\left\{\left(\frac{\epsilon^2}{p^2}\right)^\alpha\right\} - I_3, \text{ say}$$

by Lemma 1,

$$= o(1) - I_3.$$

$$I_3 = p^2.\int_\epsilon^\infty o(1)\,0\left(\frac{1}{t^3}\right)dt, \qquad \text{by Lemma 2,}$$

$$= o\left(\frac{p^2}{\epsilon^2}\right) = o(1) \qquad \text{as } \epsilon \to o.$$

This completes the proof of the theorem.

*Proof of Theorem* 2.—Theorem 2 can be proved by arguments parallel to those used in the proof of Theorem 1 with the only difference that

$$\psi_\alpha(t) = 0(1)$$

in place of

$$\psi_\alpha(t) = o(1).$$

*Proof of Theorem* 3.—Result of this theorem follows from Theorem 2 immediately.

It may be mentioned here that for the sufficiency part of Theorem 3 the following result of Prasad[3] is the most general of all the results obtained so far inasmuch as he does not presuppose any additional restriction upon $\psi(t)$.

*Theorem.*—If the integral

$$g(\theta) = \frac{1}{2\pi}\int_0^{\pi} \psi(t) \cot \tfrac{1}{2}t \, dt$$

diverges to $+\infty$ $(-\infty)$, the Abel-limit of the conjugate series (1.2) will also diverge to the same value.

I take this opportunity to express my gratitude to Dr. B. N. Prasad for his kind interest and advice in the preparation of this paper.

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# NOTE ON A THEOREM ON THE NON-SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES

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1. Recently a paper entitled "A Theorem on the non-summability of the conjugate series of a Fourier Series" has been published by U. Shukla in *Gaṇita*, December 1953, 4 (2), 95–98 but this paper does not contain any new results, much less the generalization of previous results as claimed in the paper, because the result of the author claimed as new is already known as included as a particular case in a much deeper and more general theorem published by Prasad[1] more than twenty years ago.

For, by putting

$$\frac{1}{t}\int_0^t \frac{\Psi(u)}{u}\,du = 0(1), \text{ as } t \to 0$$

in Prasad's analysis, the result of Prasad stands as:

$$V(x, \theta) - \frac{1}{4\pi}\int_\epsilon^\pi \Psi(t)\operatorname{cosec}^2 \tfrac{1}{2}t\,dt = 0(1)$$

as $x \to 1-0$, $\epsilon = \text{arc } \sin (1-x)$.

This result is more general in two directions than the result claimed to have been obtained by the author, inasmuch as whenever Shukla's condition

$$\frac{1}{t}\int_0^t \psi(u)\,du = 0(1)$$

holds, the condition of Prasad, *viz.*,

$$\frac{1}{t}\int_0^t \frac{\Psi(u)}{u}\,du = 0(1),$$

will necessarily hold, but not *vice versa*. Furthermore Prasad has used his Generalized Conjugate Function instead of the ordinary conjugate function as used by the author.

2. Further the claim of Shukla of having generalized the results of Prasad, Moursund and Anderson is not justified. For instance, for the sufficiency part Prasad's[2] is a straightforward result that the divergence to $+\infty(-\infty)$ of the conjugate function is sufficient to ensure the divergence to $+\infty(-\infty)$ of the Abel-limit of the conjugate series, whereas Shukla states this very result under an additional condition, *viz.*,

$$\frac{1}{t}\int_0^t \psi(u)\,du = 0\,(1).$$

3. Regarding the analysis used in the paper it can be easily seen by comparison that the arguments employed are *mutatis mutandis* reproduction of those used by Hardy and Rogosinski in the Proof of Theorem 76 of their Cambridge Tract " Fourier Series ". Such a result could have been mentioned by simply stating that: ' By proceeding as in the proof of Theorem 76 of Hardy and Rogosinski's book " Fourier Series " this result can be obtained.

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# STUDIES IN THE KINETICS OF THE REACTION BETWEEN HYDROGEN PEROXIDE AND POTASSIUM PERSULPHATE

## Part I. Preliminary Experiments

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Abstract

During the investigation of the reaction between potassium persulphate and potassium formate,[1] we found that $H_2O_2$ also reacts with $K_2S_2O_8$ slowly. The reaction becomes of a measurable velocity at 25° C. and above in the presence of $Ag^{\circ}$ ion as catalyst, while in the absence of any catalyst the reaction is very slow even at 40° C. Further, the reaction is found to be unimolecular with respect to persulphate ion and independent of the concentration of $H_2O_2$. Hence the total order of the reaction is approximately unity. It also depends on the concentration of $Ag^{\circ}$ ion, the reaction rate increasing with increase in the concentration of $Ag^{\circ}$ ion. The addition of $H_2SO_4$ retards the reaction. The temperature coefficient of the reaction is 1·6872 between the temperature range 25° C.–40° C., and the mean value for the energy of activation comes out to be 9659·2 cals.

The kinetics of the oxidation of several reducing agents such as iodide ion,[2] manganous ion,[3] thiosulphate ion,[4] oxalate ion,[5,6] chromium ion,[7] vanady ion,[8] ammonia,[9] ammonium ion[10] and formate ion[1] by persulphate ion have been studied. We, however, find that persulphate also reacts with $H_2O_2$ very slowly at room temperature in the absence of any catalyst and the reaction becomes fairly fast by the addition of a very small amount of $AgNO_3$ solution which acts as a very powerful catalyst [*cf.*, the oxidation of oxalate ion and other reducing agents (*loc. cit.*)].

The purpose of the present investigation is to study the mechanism of the kinetics of the hitherto unstudied reaction between persulphate ion and $H_2O_2$ and thus to find out the similarity of this reaction with other reactions involving persulphate ion. The work is being carried on with the purpose of elucidating the general mechanism of reactions in which persulphate ion is the oxidising agent.

Preliminary experiments showed that the reaction in the absence of any catalyst is very slow at ordinary temperatures and hence it was not possible to study the kinetics of this reaction in the absence of any catalyst. We, however, find that in the presence of a very small amount of $AgNO_3$ solution the reaction rate is measurable at 25° C. and above. Hence the reaction was always carried out at 30° C., unless otherwise stated, at which temperature it was found most convenient to carry out this reaction. Further, redistilled water was used in order to avoid the erratic influences of traces of impurities because we have found previously in the course of the investigation of the reaction kinetics of $K_2S_2O_8$—oxalic acid (*loc. cit.*) and $K_2S_2O_8$—HCOOK reactions (*loc. cit.*) that the reaction is generally susceptible to traces of impurities in distilled water. It was confirmed by preliminary experiments that $AgNO_3$ solution does not decompose $H_2O_2$ by itself and further it was seen that a solution of $H_2O_2$ and $K_2SO_4$ mixed to have the same ionic strength as that of the reaction mixture did not result in any perceptible decomposition of $H_2O_2$ at 40° C. even in more than 4 hours time. These experiments put aside any doubt about the reaction between $K_2S_2O_8$ and $H_2O_2$ taking place.

## EXPERIMENTAL

Potassium persulphate A.R., B.D.H. was used after recrystallisation and drying at room temperature under vacuum. $H_2O_2$ of C.P. quality was used. Other chemicals used were either of A.R. Quality or were used after recrystallisation. Standard solution of potassium persulphate was made daily by direct weighing of the salt just before the start of the experiment. The strength of the solution was verified by titration against standard ferrous sulphate solution. Standard solution of $H_2O_2$ was also made daily by titration against standard $KMnO_4$ solution.

In all the experiments redistilled water from a quartz distilling flask was used. This precaution was taken as the earlier workers on the reactions involving persulphate ion report that the reaction is generally susceptible to traces of impurities in the distilled water used and further to avoid the catalytic decomposition of $H_2O_2$ by traces of impurities.

A Jena bottle coated on the outside by black japan and wrapped in a black cloth was used as the reaction vessel because the reaction was found by us to be dependent on the intensity of light. Hence the reaction was always studied strictly in the dark to eliminate the possibility of any photochemical irregularity. Calculated amounts of $H_2O_2$ and $AgNO_3$ solution were taken in the reaction vessel kept in the thermostat maintained at the temperature of the experiment. The flask containing potassium persulphate

solution was also kept in the thermostat to attain the temperature of the bath. The reaction was started by running in the calculated quantity of potassium persulphate by means of a pipette in the reaction vessel. After suitable intervals of time 5 c.c. of the reaction mixture was withdrawn and added to ice-cold dilute $H_2SO_4$ and the remaining $H_2O_2$ was estimated against standard solution of $KMnO_4$. The titrations were done with a microburette.[1]

## Results of the Measurements

Some preliminary experiments were carried out to find whether the reaction between potassium persulphate and $H_2O_2$ proceeds firstly with measurable speed at any suitable temperature without any catalyst and secondly whether the results are reproducible. It was found that the reaction without a catalyst is very slow even at 40° C. (*vide*, Table I). A still

TABLE I

$K_2S_2O_8$ = 0·025 M; $H_2O_2$ = 0·01 M; Temp. = 40° C.

| Time in minutes | Vol. of N/50 K MnO$_4$ used in c.c. | K unimolecular $\times 10^4$ |
|---|---|---|
| 1 | 4·93 | .. |
| 10 | 4·91 | 4·350 |
| 40 | 4·87 | 3·130 |
| 80 | 4·81 | 3·119 |
| 135 | 4·73 | 3·077 |
| 180 | 4·67 | 3·024 |
| 210 | 4·62 | 3·107 |
| 245 | 4·57 | 3·105 |
| 300 | 4·50 | 3·050 |

higher temperature could not be used as slight decomposition of $H_2O_2$ by itself at higher temperature takes place. However we found that the reaction in the presence of a small amount of $AgNO_3$ has a measurable speed even at 25° C. and hence the reaction was generally studied in the presence of $AgNO_3$ as catalyst at 30° C. Further it was found that the results are

fairly reproducible and Table II gives one of the very similar results for four different sets of experiments.

TABLE II

$K_2S_2O_8 = 0{\cdot}01$ M; $H_2O_2 = 0{\cdot}01$ M; $AgNO_3 = 0{\cdot}00015$ M; Temp. = 30° C.

| Time in minutes | Vol. of N/50 $KMnO_4$ used in c.c. | K unimolecular $\times 10^2$ |
|---|---|---|
| 1 | 4·93 | .. |
| 5 | 4·66 | 1·405 |
| 10 | 4·40 | 1·261 |
| 30 | 3·32 | 1·363 |
| 40 | 2·88 | 1·375 |
| 60 | 2·26 | 1·322 |
| 81 | 1·70 | 1·331 |
| 90 | 1·52 | 1·322 |

The experimental results are presented in the tables below. The concentrations of the reactants represent the final concentrations after mixing.

Thus from the above table it is clear that in the absence of any catalyst the reaction even at a high concentration of $K_2S_2O_8$ (0·025 M) is very slow, only about 1/10th fraction decomposing in 5 hours time.

A comparison of Tables I and II reveals that the reaction which is so slow in the absence of any catalyst at 40° C. becomes of a measurable speed at a lower temperature of 30° C. and at lower concentration of $K_2S_2O_8$ by the addition of a very small quantity (0·00015 M) of $AgNO_3$. Thus as previously stated this strong catalytic effect of Ag° ion is found in almost all reactions of $S_2O_8''$ ion. The subsequent experiments were hence performed in the presence of $AgNO_3$ as catalyst. Further the constancy of the unimolecular constant at equivalent concentrations of the reactants indicates that the total order of the reaction is approximately unity.

On varying the concentration of $H_2O_2$ keeping the concentration of $K_2S_2O_8$ constant, we find that the reaction rate is only very slightly influenced by the concentration of $H_2O_2$ *vide* Table III.

TABLE III

$K_2S_2O_8$ = 0·01 M; $H_2O_2$ = 0·02 M; $AgNO_3$ = 0·00015 M, Temp. = 30° C

| Time in minutes | Vol. of N/50 $KMnO_4$ used in c.c. | K unimolecular $\times 10^2$ |
|---|---|---|
| 1 | 9·98 | |
| 5 | 9·56 | 2·199 |
| 12 | 9·20 | 1·549 |
| 15 | 9·04 | 1·494 |
| 20 | 8·76 | 1·479 |
| 30 | 8·28 | 1·440 |
| 40 | 7·84 | 1·440 |
| 50 | 7·50 | 1·407 |
| 60 | 7·16 | 1·416 |
| 80 | 6·66 | 1·391 |
| 100 | 6·36 | 1·311 |
| Infinity (calculated) | 5·00 | |

A comparison of Tables II and III clearly points out that the reaction is zero molecular with respect to $H_2O_2$. Thus the amount of $H_2O_2$ decomposed in unit time is almost independent of the initial concentration of $H_2O_2$. Further on calculating the order of the reaction with respect to $H_2O_2$, the value of $n$ in the expression

$$n = \frac{\log\left(-\frac{dc_1}{dt}\right) - \log\left(-\frac{dc_2}{dt}\right)}{\log c_1 - \log c_2}$$

comes out to be 0·0519, indicating the zero-molecularity of the reaction with respect to $H_2O_2$.

On doubling the concentration of $K_2S_2O_8$ and keeping the concentration of $H_2O_2$ constant, it is found that the velocity constant is almost doubled;

while an increase in the concentration of the catalyst $AgNO_3$ also results in an increase in the reaction velocity, *vide* Table IV.

TABLE IV

$H_2O_2 = 0 \cdot 01$ M; Temperature = 30° C.

| Concentration of $K_2S_2O_8$ | Concentration of $AgNO_3$ | K unimolecular $\times 10^2$ |
|---|---|---|
| 0·01 M | 0·00015 M | 1·339 |
| 0·02 M | 0·0002 M | 2·974 |
| 0·01 M | 0·0002 M | 1·9405 |
| 0·01 M | 0·0004 M | 3·008 |
| 0·01 M | 0·0006 M | 3·805 |

*Effect of $H_2SO_4$.*—The addition of $H_2SO_4$ retards the reaction rate slightly as will be seen from the following table:—

TABLE V

$K_2S_2O_8 = 0 \cdot 01$ M; $H_2O_2 = 0 \cdot 01$ M; $AgNO_3$ 0·0002 M.
$H_2SO_4 = 0 \cdot 1$ M. Temp. 30° C.

| Time in minutes | Vol. of N/50 $KMnO_4$ used in c.c. | K unimolecular $\times 10^2$ |
|---|---|---|
| 1 | 4·92 | .. |
| 5 | 4·60 | 1·681 |
| 10 | 4·42 | 1·193 |
| 15 | 4·14 | 1·234 |
| 20 | 3·94 | 1·169 |
| 30 | 3·55 | 1·126 |
| 41 | 3·06 | 1·1875 |
| 60 | 2·53 | 1·128 |
| 80 | 2·10 | 1·078 |
| 100 | 1·74 | 1·051 |
| 119 | 1·50 | 1·007 |

On comparing the value of unimolecular constant with the value without $H_2SO_4$ ($K = 1 \cdot 9405 \times 10^{-2}$, Table IV), we see that the reaction velocity in presence of $H_2SO_4$ is less.

The role of $H_2SO_4$ in retarding the reaction rate can only be clear after determining the salt effect.

*Effect of Temperature.*—The reaction was carried out at four different temperatures, *viz.*, 25° C., 30° C., 35° C. and 40° C., *vide* Table VI.

TABLE VI

$K_2S_2O_8 = 0{\cdot}01$ M; $H_2O_2 = 0{\cdot}01$ M; $AgNO_3 = 0{\cdot}0002$ M.

| Temperature | Unimolecular Constant × $10^2$ | Temperature Coefficeint | |
|---|---|---|---|
| 25° C. | 1·4540 | | |
| 35° C. | 2·4383 | 1·716 | mean = 1·6872 |
| 30° C. | 1·9405 | | |
| 40° C. | 3·2182 | 1·6585 | |

The energy of activation for the reaction from the above data comes out to be 9659·2 calories.

Thus it is seen that in all reactions in which persulphate ion is the oxidising agent, the concentration of the reductant does not influence the reaction rate.

Further investigations in this reaction to determine the nature of the salt effect and effect of different catalytic agents in order to elucidate the mechanism of the reaction are in progress.

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# THE EFFECT OF SURFACE-ACTIVE SUBSTANCES ON THE REDOXOKINETIC EFFECT

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## Introduction

It is well known that surface-active substances modify electrode processes taking place at metal aqueous solution interfaces. Work in this laboratory has given similar effects on the current maxima obtained by the Breyer Gutmann technique. It was, therefore, considered of interest to study the effect of surface-active substances on the 'redoxokinetic effect'.

## Experimental

The experimental set-up for measuring the redoxokinetic potential is same as was used earlier. The surface-active substances used were Amyl Alcohol, Thymol Blue, Methyl Red and Albumin. The first was added to the extent of 20% by volume while the second to the extent that it gave a concentration of ·002%. The other substances were added to give a concentration of ·004% in each case. The redox system used was ferrous-ferric containing 0·25 M ferrous ammonium sulphate and 0·25 M ferric sulphate.

## Discussion

An examination of plot between the redoxokinetic potent and corresponding current values shows that the addition of Amyl Alcohol produces but little change in the redoxokinetic potential. Thymol Blue and Methyl Red produce the same effect within the limits of the experimental error. Albumin produces a very large change in the redoxokinetic potential at corresponding current values. In every case where there is a change the surface-active substance has enhanced the 'redoxokinetic effect'. This is presumably due to the surface active substance getting adsorbed and rendering that part of the surface unavailable for electrode processes. At corresponding current values, therefore, the electrode having an adsorption film of surface-active substance would affectively have a higher current density. As the whole of the current passing would have to be concentrated at the bare portions of the surface, it appears Amyl Alcohol is not very much taken up by the platinum surface; Thymol Blue and Methyl Red are adsorbed to an intermediate extent and Albumin is strongly adsorbed.

The increase in the effective current density due to the presence of the adsorption film, increases the amplitude of the alternating potential variations at the micro-platinum electrode. This can be seen from the values of a.c. potential of the electrodes as measured by the Du Mont Oscillograph. The change in the redoxokinetic potential consequent on the addition of the surface-active substance appears to be mainly due to the change in the amplitude of the a.c. potential at the electrode.

In other words, the redoxokinetic potential is perhaps practically a single-value function of the a.c. potential at the micro-electrode. In the curve in which the redoxokinetic potential values obtained in all the systems studied herein are plotted against the corresponding a.c. potential values at the micro-electrode; most of the points appear to fall along a single curve, although albumin appears to show some individuality.

# ON THE DERIVATIVES OF $\left(\frac{\omega}{e^{\omega}-1}\right)^k$ AT $\omega = 0$

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THEOREM.—

$$\left[\frac{d^r}{d\omega^r}\left(\frac{\omega}{e^{\omega}-1}\right)^k\right]_{\omega=0} = (-1)^r \frac{\Sigma\, a_1 a_2 \ldots . a_r}{{}^{k-1}C_r} \tag{1}$$

in which $k$ is any given positive integer $\geqslant 2$, $1 \leqslant r \leqslant k-1$; and where $\Sigma\, a_1, a_2, \ldots, a_r$ denotes the sum of the $\binom{k-1}{r}$ products of $r$ factors each formed by taking the possible combinations of the $k-1$ quantities $k-1$, $k-2, \ldots, 2, 1$, $r$ at a time.

*Proof.*—To establish this theorem it is sufficient, after defining the right member of (1) to be unity in case $r = 0$, to verify the following identity in $x$ where $x$ is regarded as an auxiliary variable:

$$\sum_{r=0}^{k-1}\left[\frac{(k-1)\,!}{r\,!\,(k-r-1)!}\left\{\frac{d^r}{d\omega^r}\left(\frac{\omega}{e^{\omega}-1}\right)^k\right\}_{\omega=0} x^{k-r-1}\right]$$

$$\equiv \sum_{r=0}^{k-1}\left\{(-1)^r\,\Sigma\, a_1, a_2, \ldots, a_r\right\} x^{k-r-1}. \tag{2}$$

This is equivalent to (1) because the coefficients of like powers of $x$ on the right and on the left sides of (2) will necessarily be equal if (2) is an identity and the equating of these coefficients gives precisely (1) wherein $r$ ranges over the prescribed range; namely, $1 \leqslant r \leqslant k-1$.

Inasmuch as (2) is readily found by trial to be true in case $k$ is small, as $k = 2$ and $k = 3$, the inductive process of proof will be employed.

Assuming then the correctness of (2) for any particular $k$, we proceed to show that it holds true also when $k$ is replaced by $k+1$. To do this let us first multiply both members of (2) by $(x-k)$. The product of the left member by $k$ is evidently

$$\sum_{r=0}^{k-1}\left\{\frac{-k\,(k-1)\,!}{r\,!\,(k-r-1)\,!}\left[\frac{d^r}{d\omega^r}\left(\frac{\omega}{e^{\omega}-1}\right)^k\right]_{\omega=0}\right\} x^{k-r-1} \tag{3}$$

Before multiplying the same member of (2) also by $x$ it is desirable to write it in the form

$$x^{k-1} + \sum_{r=0}^{k-1} \left\{ \frac{(k-1)\,!\,(k-r-1)}{(r+1)\,!\,(k-r-1)\,!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^{\omega}-1} \right)^{k} \right]_{\omega=0} x^{k-r-2} \right\} \tag{4}$$

The product of (4) and $x$ therefore becomes

$$x^{k} + \sum_{r=0}^{k-1} \left\{ \frac{(k-1)\,!\,(k-r-1)}{(r+1)\,!\,(k-r-1)\,!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^{\omega}-1} \right)^{k} \right]_{\omega=0} x^{k-r-1} \right\} \tag{5}$$

In considering the product of the right members of (2) and $(x-k)$, it is necessary to recall the interpretation of $\Sigma\, a_1, a_2, \ldots, a_r$ as originally given in the statement of the theorem. If we employ the root coefficient relations of the elementary theory of equations the right member of (2) is seen to be

$$(x-k-1)\,(x-k-2)\ldots\ldots(x-2)\,(x-1).$$

Hence the product of this right member by $(x-k)$ is

$$(x-k)\,(x-k-1)\ldots\ldots(x-2)\,(x-1) \tag{6}$$

which may be expressed in the form

$$x^{k} + \sum_{r=0}^{k-1} \{(-1)^{r+1}\, \Sigma\, a_1, a_2, \ldots, a_{r+1}\}\, x^{k-r-1}, \tag{7}$$

where the groups of factors employed in $\Sigma\, a_1, a_2, \ldots\, a_{r+1}$ are now all the possible combinations of the quantities $k, k-1, \ldots, 2, 1$ taken $(r+1)$ at a time. Hence, having assumed the truth of (2) for any given value of $k$, we have the following identity in $x$:

$$x^{k} + \sum_{r=0}^{k-1} \left\{ \frac{-k\,(k-1)\,!}{r\,!\,r\,(k-r-1)\,!} \left[ \frac{d^{r}}{d\omega^{r}} \left( \frac{\omega}{e^{\omega}-1} \right)^{k} \right]_{\omega=0} \right. \tag{8}$$

$$\left. + \frac{(k-1)\,!\,(k-r-1)}{(r+1)\,!\,(k-r-1)\,!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^{\omega}-1} \right)^{k} \right]_{\omega=0} \right\} x^{k-r-1}$$

$$= x^{k} + \sum_{r=0}^{k-1} \{(-1)^{r+1}\, \Sigma\, a_1, a_2, \ldots, a_{r+1}\}\, x^{k-r-1}$$

But we may now show that the left member of (8) is identically equal to

$$x^k + \sum_{r=0}^{k-1} \left\{ \frac{k\,!}{(r+1)\,!\,(k-r-1)\,!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^\omega - 1} \right)^{k+1} \right]_{\omega=0} x^{k-r-1} \right\} \tag{9}$$

which statement is equivalent to showing that (2) holds true, as desired when $k$ is replaced by $(k+1)$ in as much as (8) will then have the form

$$\sum_{r=0}^{k} \left\{ \frac{k\,!}{r\,!\,(k-r)\,!} \left[ \frac{d^r}{d\omega^r} \left( \frac{\omega}{e^\omega - 1} \right)^{k+1} \right] x^{k-r} \right\}$$

$$= \sum_{r=0}^{k} [(-1)^r \,\Sigma\, a_1, a_2, \ldots, a_r]\, x^{k-r}$$

which is what results from (2) when $k$ is replaced by $(k+1)$. To do this, it suffices to show that the corresponding coefficients of the like powers of $x$ in (9) and in the left member of (8) are equal. In other words, it is sufficient to show

$$\frac{-k\,(k-1)\,!}{r\,!\,(k-r-1)\,!} \left[ \frac{d^r}{d\omega^r} \left( \frac{\omega}{e^\omega - 1} \right)^k \right]_{\omega=0}$$

$$+ \frac{(k-1)\,!\,(k-r-1)}{(r+1)\,!\,(k-r-1)\,!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^\omega - 1} \right)^k \right]_{\omega=0}$$

$$= \frac{k\,!}{(r+1)\,!\,(k-r-1)!} \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^\omega - 1} \right)^{k+1} \right]_{\omega=0} \tag{10}$$

Now, let us assume for the moment the correctness of (10). If we multiply both its members by $\dfrac{(r+1)\,!\,(k-r-1)\,!}{(k-1)\,!}$ we obtain the equivalent relation

$$(k-r-1) \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^\omega - 1} \right)^k \right]_{\omega=0}$$

$$- k\,(r+1) \left[ \frac{d^r}{d\omega^r} \left( \frac{\omega}{e^\omega - 1} \right)^k \right]_{\omega=0}$$

$$- k \left[ \frac{d^{r+1}}{d\omega^{r+1}} \left( \frac{\omega}{e^\omega - 1} \right)^{k+1} \right]_{\omega=0} = 0 \tag{11}$$

For convenience, let us now write $k$ in the form $k = -p, p \leq -2$. Then we may rewrite (11) in the form

$$-(p+r+1)\left[\frac{d^{r+1}}{d\omega^{r+1}}\left(\frac{\omega}{e^\omega-1}\right)^{-p}\right]_{\omega=0}$$

$$+p(r+1)\left[\frac{d^r}{d\omega^r}\left(\frac{\omega}{e^\omega-1}\right)^{-p}\right]_{\omega=0}$$

$$+p\left[\frac{d^{r+1}}{d\omega^{r+1}}\left(\frac{\omega}{e^\omega-1}\right)^{-p-1}\right]_{\omega=0} = 0 \qquad (12)$$

If $p$ now be treated temporarily as a variable, it is a matter of observation to see that the left member of (12) is a rational polynomial in $p$ of degree $r+2$. Hence, if it can be shown that the algebraic equation (12) is satisfied by every positive integer $p$, its identical nature for all real values of $p$ is established and hence the identical nature of (11) is established for all real values of $k$; in particular, for the given positive integral value with which we are concerned. In order to do this, it is desirable at this point to recall that

$$\left[\frac{d^r}{d\omega^r}\left(\frac{e^\omega-1}{\omega}\right)^p\right]_{\omega=0} = \frac{r\,!}{(r+p)\,!}\left[\frac{d^{r+p}}{d\omega^{r+p}}(e^\omega-1)^p\right]_{\omega=0}, \qquad (13)$$

where $p$ is any positive integer.

This follows immediately from a consideration of

$$\frac{d^{r+p}}{d\omega^{r+p}}\left[\omega^p\left(\frac{e^\omega-1}{\omega}\right)^p\right]_{\omega=0}$$

In fact, if we employ the theorem of Leibnitz upon the derivative of the product within the bracket, it is seen that the substitution of $\omega=0$ causes all derivatives of $\omega^p$ to vanish except the $p$th. The term corresponding to this derivative is

$$\frac{(r+p)\,!}{r\,!}\left[\frac{d^r}{d\omega^r}\left(\frac{e^\omega-1}{\omega}\right)^p\right]_{\omega=0}.$$

Relation (13) then follows immediately.

Employing (13) upon each term of (12) we get

$$-\left[\frac{d^{r+p+1}}{d\omega^{r+p+1}}(e^\omega-1)^p\right]_{\omega=0}+p\left[\frac{d^{r+p}}{d\omega^{r+p}}(e^\omega-1)^p\right]_{\omega=0}$$

$$+p\left[\frac{d^{r+p}}{d\omega^{r+p}}(e^\omega-1)^{p-1}\right]_{\omega=0} = 0. \qquad (14)$$

Now the first term of (14) may be written as

$$-\left[\frac{d^{r+p}}{d\omega^{r+p}}\left(\frac{d}{d\omega}(e^{\omega}-1)^{p}\right)\right]_{\omega=0} \tag{15}$$

which becomes after actually obtaining the derivative involved

$$-p\left[\frac{d^{r+p}}{d\omega^{r+p}}(e^{\omega}-1)^{p-1}\right]_{\omega=0}-p\left[\frac{d^{r+p}}{d\omega^{r+p}}(e^{\omega}-1)^{p}\right]_{\omega=0} \tag{16}$$

The substitution of this result in (14) causes it to vanish identically. It is now readily observed that it is possible to retrace our steps from this identity to establish the validity of (10).

We have thus established the equality of the corresponding coefficients of like powers of $x$ in (9) and in the left member of (8). Thus (2) also holds true when $k$ is replaced by $(k+1)$. Hence the induction is complete and the theorem is established.

My thanks are due to Dr. P. L. Srivastava under whose guidance the result is obtained.

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# AN EXTENSION OF A RESULT IN FACTORIAL SERIES

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1. G. T. Miller and H. K. Hughes[1] have obtained expression for the factorial series

$$\sum_{m=0}^{\infty} \frac{g(m)(-1)^{m(k+1)}}{z(z+1)\ldots\ldots(z+m)} \tag{1.1}$$

where $g(m)$ satisfies certain conditions. This expression defines the function represented by the series throughout the whole $z$-plane excepting at $z=0$, $-1$, $-2, \ldots\ldots$ In this paper similar expression has been obtained for a more general type of the above series, where the coefficient $(-1)^{m(k+1)}$ occurring in the series is replaced by $e^{Ai(k+1)m}$ $(0 \leqslant A \leqslant 2\pi)$. In theorem (I) an expression for (1.1) has been obtained under more general conditions on the order of $g(m)$.

2. Theorem I. Given the factorial series

$$\Omega(z) = \sum_{m=0}^{\infty} \frac{e^{Aim(k+1)} g(m)}{z(z+1)\ldots(z+m)}, \quad (0 < A \leqslant 2\pi) \tag{2.1}$$

whose abscissa of convergence is finite and for which $k$ is a positive integer, let it be assumed that the function $g(m)$ occurring in the coefficient of (2.1) when considered as a function $g(\omega)$ of the complex variable $\omega = x + iy$, satisfies the following conditions:

(*a*) It is single-valued and analytic throughout the half-plane $x \geqslant -\frac{1}{2}$.

(*b*) For all $x \geqslant -\frac{1}{2}$ and for all values of $|y|$ sufficiently large, we have:

$$\left|\frac{g(x+iy)}{g(x)}\right| < M_\epsilon \exp\left[(k+1)A - \frac{\pi}{2} - \epsilon\right]|y| \tag{2.2}$$

$$\left|\frac{g(x-iy)}{g(x)}\right| < M_\epsilon' \exp\left[2\pi(k+1) - \frac{\pi}{2} - A(k+1) - \epsilon\right]|y|$$

($M_\epsilon$ and $M_\epsilon'$ being constants).

Moreover let there exist a positive constant $q$, dependent on $\epsilon$ but independent of $z$, such that when $z$ is confined to any finite region, we have,

$$\left|\frac{g(x)}{\Gamma(z+x+1)}\right| < \frac{N}{x^{1+\epsilon}} \tag{2.3}$$

when $x > q$ and where $N$ is a positive constant.

Then the function $\Omega(z)$ defined by (2.1) may be continued analytically throughout the whole finite plane except in the neighbourhood of the points $z = 0, -1, -2, \ldots$ and throughout this region will be defined by the equation

$$\Omega(z) = (-1)^{k+1}\,\Gamma(z)\int_{-\frac{1}{2}}^{\infty}\frac{(1-\cos 2\pi kx + i\sin 2\pi kx)\,g(x)e^{Ai(k+1)x}}{(\cos 2\pi x + i\sin 2\pi x - 1)\,\Gamma(z+x+1)}\,dx$$

$$+ (-1)^{k}\,\Gamma(z)\,2k\pi i\int_{-\infty}^{\infty}\frac{F(y,z)}{(e^{-2\pi y}+1)^{k+1}}\,dy \tag{2.4}$$

where

$$F(y,z) = \int_{0}^{y}\frac{(e^{2\pi(s-y)}-1)^{k-1}\,e^{2\pi(s-y)}\,g(-\frac{1}{2}+is)\,e^{(-\frac{1}{2}+is)Ai(k+1)}}{\Gamma(z+\frac{1}{2}+is)}\,ds \tag{2.5}$$

*Proof of Theorem.*—The proof of this theorem is based upon the following statement which is seen to be a consequence of Cauchy's integral theorem in the calculus of residues, namely, if $P(\omega)$ and $Q(\omega)$ are any two functions of the complex variable $\omega = x + iy$ both of which are single valued and analytic throughout a region $A$ of the $\omega$-plane and of which $Q(\omega)$ vanishes within $A$ only at the points $\omega = \lambda_1, \lambda_2, \ldots, \lambda_m$ which are zeroes of the first order and if $c$ denotes any closed contour lying within $A$ and including the points $\omega = \lambda_1, \lambda_2, \ldots, \lambda_m$, then one may write

$$\frac{1}{2\pi i}\int_{c}\frac{P(\omega)}{[Q(\omega)]^{k+1}}\,d\omega = \sum_{m=1}^{m}\frac{1}{k!}\left\{\frac{d^k}{d\omega^k}\left[P(\omega)\left(\frac{\omega-\lambda_n}{Q(\omega)}\right)^{k+1}\right]\right\}_{\omega=\lambda_m}$$

in which $k$ denotes any positive integer. The general term of the summation upon the right is the residue at $\omega = \lambda_m$ of the integrand upon the left as may be readily verified by observation of its Laurent's expansions.

If the formula of Leibnitz be employed upon the derivative of the product within the bracket, it follows that

$$\frac{1}{2\pi i}\int_{c}\frac{P(\omega)}{\{Q(\omega)\}^{k+1}}\,d\omega=\sum_{m=1}^{m}\frac{1}{k!}\sum_{t=0}^{k}\frac{k!}{t!\,(k-t)!}\left[\frac{d^{k-t}}{d\omega^{k-t}}P(\omega)\right]_{\omega=\lambda_m}\left[\frac{d^t}{d\omega^t}\left(\frac{\omega-\lambda_m}{Q(\omega)}\right)^{k+1}\right]_{\omega=\lambda_m}$$

For the proof of the theorem of this paper, we shall choose $Q(\omega)=(e^{2\pi i\omega}-1)$. With this choice, $\lambda_m=m$ where $m$ is zero or any positive integer. Through the medium of elementary transformations it is found that

$$\left[\frac{d^t}{d\omega^t}\left(\frac{\omega-m}{e^{2\pi i\omega}-1}\right)^{k+1}\right]_{\omega=m}=\frac{(2\pi i)^t}{(2\pi i)^{k+1}}\left[\frac{d^t}{d\omega^t}\left(\frac{\omega}{e^{\omega}-1}\right)^{k+1}\right]_{\omega=0}$$

Hence it follows at once that

$$\left[\frac{d^t}{d\omega^t}\left(\frac{\omega-m}{e^{2\pi i\omega}-1}\right)^{k+1}\right]_{\omega=m}=\frac{(2\pi i)^t\,(-1)^t\,(k-t)!\,t!}{(2\pi i)^{k+1}\,k!}\sum a_1, a_2,\ldots, a_t$$

where $\Sigma\, a_1\, a_2, \ldots, a_t$ denotes the sum of the $(k/t)$ products of $t$ factors each formed by taking the possible combinations of the $k$ quantities $k, (k-1), (k-2), \ldots, 2, 1$ $t$ at a time. Hence we have

$$\frac{1}{2\pi i}\int_{c}\frac{P(\omega)}{(e^{2\pi i\omega}-1)^{k+1}}\,d\omega=\sum_{m=0}^{m}\frac{1}{k!\,(2\pi i)^{k+1}}\sum_{t=0}^{k}(-1)^t\,(2\pi i)^t\,\{\Sigma a_1, a_2,\ \ldots, a_t\}\,P^{(k-t)}(\omega) \qquad (2.6)$$

where $\Sigma\, a, a_2 \ldots, a_t$ is defined as above and the integration is performed in the positive sense.

Let us suppose that $z$ has any fixed value which is real and positive and greater than $\lambda$, the abscissa of convergence of the series (2.1). We shall write the series (2.1) in the form

$$\Omega(z)=\sum_{m=0}^{\infty}\frac{\Gamma(z)\,g(m)\,e^{Ai(k+1)m}}{\Gamma(z+m+1)} \qquad (2.7)$$

and shall seek to determine $P(\omega)$ so that

$$\sum_{t=0}^{k}(-1)^t\,(2\pi i)^t\,(\Sigma a_1, a_2, \ldots, a_t)\,P^{(k-t)}(m)=\frac{g(m)\,e^{Ai(k+1)m}}{\Gamma(z+m+1)} \qquad (2.8)$$

This is a differential equation in $m$ with constant coefficients. The roots of the characteristic equation are distinct and are $2\pi ik$, $2\pi i(k-1)$, ...., $2\pi i$. If we denote these roots by $2\pi ir_j$ where

$$r_j = k+1-j,\ 1 \leqslant j \leqslant k; \tag{2.9}$$

then a particular solution of (2.8) is given by

$$P(m) = \sum_{j=1}^{k} b_j \exp r_j 2\pi im \int_0^m \frac{\exp(-r_j 2\pi iy)\, g(y)\, e^{Ai(k+1)y}}{\Gamma(z+y+1)} dy \tag{2.10}$$

where

$$b_j = \frac{(-1)^{j-1}}{(2\pi i)^{k-1}} \cdot \frac{1}{(j-1)!\,(k-j)!} \tag{2.11}$$

If we replace $m$ by the complex variable $\omega = x + iy$, then $P(\omega)$ is an analytic function of $\omega$ in the finite $\omega$-plane. Hence if $P(\omega)$ has the form indicated by (2.10), then the equation (2.6) can be written in the form

$$\sum_{m=0}^{n} \frac{e^{Ai(k+1)m}\, g(m)\, \Gamma(z)}{\Gamma(z+m+1)} = \Gamma(z)\,(2\pi i)^k\, k! \int_c \frac{P(\omega)}{(e^{2\pi i\omega}-1)^{k+1}} d\omega \tag{2.12}$$

where the contour $c$ is so chosen as to enclose the points $\omega = 0, 1, 2, \ldots, n$, but no other vanishing points of $(e^{2\pi i\omega} - 1)$.

If we substitute the form for $P(\omega)$ from (2.10) in (2.12) we obtain a contour integral for the determination of the first $(n+1)$ terms of the series (2.7) as follows:

$$\sum_{m=0}^{n} \frac{e^{Ai(k+1)m}\, g(m)\, \Gamma(z)}{\Gamma(z+m+1)} = \Gamma(z)\,(2\pi i)^k\, k!$$

$$\sum_{j=1}^{k} b_j \int \frac{e^{r_j 2\pi i\omega}\, I(\omega, z)}{(e^{2\pi i\omega}-1)^{k+1}} d\omega \tag{2.13}$$

where

$$I(\omega, z) = \int_0^{\omega} \frac{\exp(-r_j 2\pi i\omega + Aik+1\omega)\, g(\omega)}{\Gamma(z+\omega+1)} d\omega \tag{2.14}$$

and where $b_j$ is given by (2.11).

We shall choose as the path of integration the rectangle enclosed by the ines (1) $\omega = x + ip$, (2) $\omega = -\frac{1}{2} + iy$, (3) $\omega = x - ip$ and (4) $\omega = n + \frac{1}{2} + iy$, and shall denote the integrals along these sides by $A$, $B$, $D$ and $E$ respectively. Then in order to evaluate the contour integral

$$\int_c \frac{P(\omega)}{(e^{2\pi i\omega} - 1)^{k+1}}\, d\omega$$

we must consider $k$ integrals of the form

$$\int_c \frac{\exp r_j\, 2\pi i\omega\, I(\omega, z)}{(e^{2\pi i\omega} - 1)^{k+1}}\, d\omega \tag{2.15}$$

where $1 \leqslant j \leqslant k$ and $I(\omega, z)$ is given by (2.14).

In (2.14) let us take the path of integration along the $x$ axis from $\omega = 0$ to $\omega = x$ and then along a line parallel to the $y$-axis to the point $\omega = x + iy$. The function $I(\omega, z)$ then can be written in the form

$$I(\omega, z) = R(x, z) + iS(x, y, z),$$

where

$$R(x, z) = \int_0^x \frac{\exp(-r_j\, 2\pi i x + Aik + 1\, x)\, g(x)}{\Gamma(z + x + 1)}\, dx$$

and

$$S(x, y, z) = \int_0^y \frac{\exp(-r_j\, 2\pi i\, \overline{x + iy} + Ai\, \overline{k+1}\, \overline{x + iy})\, g(x + iy)}{\Gamma(z + x + 1 + iy)}\, dy$$

With regard to $R(x, z)$ it is sufficient to note that $|R(x, z)|$ is bounded when $x$ is bounded.

It is known that for all $x > x_0$ where $x_0$ is arbitrary, and for all $y$ sufficiently large, we have

$$\left|\frac{1}{\Gamma(x + iy)}\right| < K \exp\left[\tfrac{1}{2}(\pi + \epsilon)\,|y|\right] \tag{2.16}$$

From the above inequality, together with (2.2) and (2.9) it follows that the absolute value of the integrand in (2.14) is less than

$$m_1 \exp\left[(r_j 2\pi y - \frac{\epsilon}{2} y\right], \quad y \to +\infty$$

is less than

$$m_2 \exp\left[r_j 2\pi y + 2\pi\,\overline{k+1}\,|y| - \frac{\epsilon}{2}|y|\right], \quad y \to -\infty$$

the constants $m_1$ and $m_2$ being dependent only on $\epsilon$. It follows that there exists a constant $K$ dependent upon $\epsilon$, such that

$$|S(x, y, z)| < K \exp\left(2\pi r_j y - \frac{\epsilon}{2} y\right), \quad y \to \infty \tag{2.17}$$

and

$$|S(x, y, z)| < K \exp\left(2\pi y r_j + 2\pi\,\overline{k+1}\,|y| - \frac{\epsilon}{2}|y|\right), \quad y \to -\infty.$$

Consider now the various contributions to the integral (2.15). We shall first consider the contribution $A$. Here we have $\omega = x + ip$.

In order to evaluate $A$ we must consider $k$ integrals of the form

$$A_j = \int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{\exp r_j 2\pi i (x+ip)\,[R(x,z) + iS(x,p,z)]}{(e^{2\pi i \overline{x+ip}} - 1)^{k+1}}\, dx$$

where $1 \leqslant j \leqslant k$.

It is evident that we may write

$$|A_j| \leqslant \left|\int_{n+\frac{1}{2}}^{-\frac{1}{2}} \frac{R(x,z)}{e^{2p\pi r_j}}\, dx\right| + \left|\int_{n+\frac{1}{2}}^{-\frac{1}{2}} e^{-2p\pi r_j} S(x,p,z)\, dx\right|. \tag{2.18}$$

Since $R(x, z)$ is bounded if $x$ is bounded the first integral on the right of (2.18) approaches zero as $p$ approaches infinity.

With regard to the second integral on the right of (2.18) we have from (2.17) that

$$|S(x, p, z)| < K \exp\left(2\pi r_j p - \frac{\epsilon}{2} p\right).$$

The absolute value of the integrand in the integral in question is therefore of the order of $\exp(-\frac{1}{2}\epsilon p)$ for $p$ large and since $\epsilon$ is positive we conclude that $\lim\limits_{p\to\infty} A_j = 0$.

In a similar manner it may be shown that $\dim\limits_{p\to\infty} D = 0$.

We next consider the contribution $E$. $E$ consists of $k$ integrals of the form

$$E_j = (-1)^{r_j} \left[ iR(n+\tfrac{1}{2}, z) \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)}{(-e^{-2\pi y}-1)^{k+1}}\, dy \right]$$

$$+ (-1)^{r_j+1} \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)\, S(n+\tfrac{1}{2}, y, z)}{(-e^{-2\pi y}-1)^{k+1}}\, dy$$

We shall set

$$\int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)}{(-e^{-2\pi y}-1)^{k+1}}\, dy = H_j.$$

Then we can write

$$H_j = \int_{-\infty}^{\infty} (-1)^{k+1} \frac{\exp(-r\, 2\pi y)}{(e^{-2\pi y}+1)^{k+1}}\, dy = (-1)^{k+1} \cdot \frac{1}{2\pi} \cdot \frac{(k-j)!(j-1)!}{k!}.$$

Let us consider the second integral on the right of (2.19). The integral can be written:

$$\int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)\, S(n+\tfrac{1}{2}, y, z)}{(-e^{-2\pi y}-1)^{k+1}}\, dy$$

$$= \frac{g(m)}{\Gamma(z+m+1)} \int_{-\infty}^{\infty} \frac{\exp(-r_j 2\pi y)\, S_1(n+\tfrac{1}{2}, y, z)}{(-e^{-2\pi y}-1)^{k+1}}\, dy \qquad (2.20)$$

where

$$S_1(n+\tfrac{1}{2}, y, z) = \frac{\Gamma(z+n+1)}{g(n)}\, S(n+\tfrac{1}{2}, y, z)$$

$$= e^{Ai(n+\frac{1}{2})(k+1)-i2\pi r_j(n+\frac{1}{2})} \int_0^y \left[ \frac{\Gamma(z+n+1)\, g(n+\frac{1}{2}+iy)}{\Gamma(z+n+\frac{3}{2}+iy)\, g(n)} \times e^{-A(k+1)y+r_j 2\pi y} \right] dy.$$

We now make use of the following well known property of the gamma function

$$\Gamma(x+iy) = (2\pi)^{\frac{1}{2}} (x+iy)^{x-\frac{1}{2}} \exp\left(-x - y \tan^{-1} \frac{y}{x}\right)(1+\delta)$$

Where $\delta$ approaches zero as either $x$ or $y$ becomes infinite. Then we have

$$\left|\frac{\Gamma(z+n+1)}{\Gamma(z+n+\frac{3}{2}+iy)}\right| < K \exp\left(\frac{\pi}{2}|y|\right) \tag{2.21}$$

By the use of (2.21) and (2.2) and taking into account the behaviour of $(-e^{-2\pi y}-1)$ for $|y|$ large, it follows that the integrand of the right member of (2.20) is of the order of $\exp(-\epsilon|y|)$ for $|y|$ large and hence the integral converges.

Furthermore, since the series (2.1) was assumed convergent, we have

$$\lim_{n\to\infty} \frac{g(n)}{\Gamma(z+n+1)} = 0$$

Consequently the integral on the left of (2.20) tends to zero as $n$ approaches infinity. The total contribution $E_j$ is given by

$$E_j = (-1)^{r_j}\, i\, H_j \lim_{n\to\infty} R(n+\tfrac{1}{2}, z).$$

We next consider the contribution $B$ arising from the integration along the side where $\omega = -\frac{1}{2}+iy$ and we must consider $k$ integrals of the form:

$$B_j = \int_{\infty}^{-\infty} \frac{i\exp\{r_j 2\pi i(-\frac{1}{2}+iy)\}\, R(-\frac{1}{2}, z)}{(-e^{-2\pi y}-1)^{k+1}}\, dy \tag{2.22}$$

$$+\, i\int_{\infty}^{-\infty} \frac{i\exp\{r_j 2\pi i(-\frac{1}{2}+iy\}\, S(-\frac{1}{2}, y, z)}{(-e^{-2\pi y}-1)^{k+1}}\, dy$$

The first of the integrals above when combined with the contribution $E_j$ can be written in the form:

$$(-1)^{r_j}\, i\, H_j \lim_{n\to\infty} [R(n+\tfrac{1}{2}, z) - R(-\tfrac{1}{2}, z)]$$

$$= i(-1)^{r_j} H_j \int_{-\frac{1}{2}}^{\infty} \frac{g(x)\exp(-r_j 2\pi i x + Ai\,\overline{k+1}\,x)}{\Gamma(z+x+1)}\, dx$$

If we multiply the above integral by $b_j$, we obtain as the sum of the $k$ integrals in question when substituted in (2.13)

$$(-1)^{k+1}\Gamma(z)\int_{-\frac{1}{2}}^{\infty} \frac{1-\cos 2\pi kx + i\,\mathrm{Sin}\, 2\pi kx}{\cos 2\pi x - 1 + i\sin 2\pi x} \times \frac{e^{Ai(k+1)x}\, g(x)}{\Gamma(z+x+1)}\, dx \tag{2.23}$$

Finally let us investigate the second integral in (2.22). When substituted in (2.13) the sum of the $k$ integrals in question may be written in the form

$$\Gamma(z)\, 2\pi ik\, (-1)^k \int_{\infty}^{-\infty} \frac{\displaystyle\sum_{j=1}^{k} \int_0^y \frac{e^{r_j 2\pi(s-y)+\lambda i\overline{k+1}(-\frac{1}{2}+is)}(-1)^{k-j}\binom{k-1}{j-1} g(-\frac{1}{2}+is)}{\Gamma(z+\frac{1}{2}+is)}\, ds}{(e^{-2\pi y}+1)^{k+1}}\, dy \tag{2.24}$$

But

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} \exp 2\pi r_j (s-y) = e^{2\pi(s-y)} \times [e^{2\pi(s-y)} - 1]^{k-1} \tag{2.25}$$

Upon substituting (2.25) in (2.24) we find the value of this contribution to be

$$\Gamma(z)\, 2\pi ik\, (-1)^k \int_{+\infty}^{-\infty} \frac{\int_0^y [e^{2\pi(s-y)}\{e^{2\pi(s-y)} - 1\}^{k-1} g(-\frac{1}{2}+iS)]/\Gamma(z+\frac{1}{2}+is)\, ds}{(e^{-2\pi y}+1)^{k+1}}\, dy \tag{2.26}$$

The value of the right member of (2.13) is therefore the sum of the expressions (2.23) and (2.26). We thus obtain the equation (2.4).

Up to this point we have restricted $z$ to values which are real, positive and greater than $\lambda$, the abscissa of convergence, $\Omega(z)$ as defined by the series (2.1) is a function which is analytic throughout the half-plane $R(z) > \lambda$ with the exception of the points $z = 0, -1, -2, \ldots\ldots$ The right member of (2.4) furnishes an analytic continuation of $\Omega(z)$ in any finite region which does not contain the points $z = 0, -1, -2, \ldots.$ since throughout such a region the member under discussion is an analytic function. To prove this last statement, it is sufficient to show that the integral on the right of (2.4) converges uniformly since it is dominated by $\int^{\infty} x^{-1-\epsilon}\, dx$.

From (2.2), (2.5) and (2.16) it is evident that the second integral converges uniformly since it is dominated by

$$\int^{\infty} e^{-2\pi ky}\, dy \qquad \text{as } y \to +\infty$$

and by

$$\int^{\infty} e^{-(k+1)2\pi|y|+2\pi k|y|}\, dy \qquad \text{as } y \to -\infty.$$

The theorem is therefore established.

## EXTENSIONS OF THE THEOREM

In the hypatheses of the theorem it was required that the function $g(\omega)$ be analytic for all values of $\omega = x + iy$ for which $x \geqslant -\frac{1}{2}$. This condition is unnecessarily restrictive. To take up the case when $g(\omega)$ has no other singularities than a finite number of poles at the points $\omega = \lambda_1, \lambda_2, \ldots, \lambda_p$; ($\lambda_t \neq$ an integer, and $R(\lambda_t) \neq -\frac{1}{2}$); we need only observe that (2.4) remains true provided we add to the left member the expression

$\sum_1^p r_t$, where $r_t$ is the residue of the function

$$\frac{\Gamma(z)\, g(\omega)\, e^{Ai(k+1)\omega}}{\Gamma(z+\omega+1)\,(e^{2\pi i\omega}-1)^{k+1}}$$

at the point $\omega = \lambda_t$.

It is obvious that the line $\omega = -\frac{1}{2}+iy$ of the contour $c$ can be replaced by any line $\omega = k + iy$, where $-1 < k < 0$ provided in (2.1) (*a*) the half-plane is at least $R(\omega) \geqslant k$. Further, if the series (2.1) converges for every value of $x$, then $\Omega(z)$ is defined by (2.4) at all points of the $z$ plane. In case $g(\omega)$ has a finite number of branch points we need only add to the left member of (2.4) the sum of the loop integrals at these points.

I wish to thank Dr. P. L. Srivastava for his many useful suggestions.

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